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Regular Graph Representation Theorems

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Abstract

The networking theory developed in this article applies to the initial design phase of networks from diverse fields of interest. For example, computer hardware networks, communications networks, circuitry, and logistics systems all give rise to networking applications. Recently logistics engineers have realized the potential for enhancing logistics support by using such techniques. In most applications of network theory, mathematical programming and algorithmic methods are combined with network theory design to determine optimal flow and/or minimized costs. This article focuses in on the invariant aspects of what makes a network design, rather than optimization aspects. Applications of the theory developed occur in the early planning stages of a new logistics system. This is in keeping with the spirit of logistics engineering planning for large sized projects. This article bases itself on the premise that early holistic planning prevents costly post deployment cures.

Investigating the simple regular graph might seem to be a narrow pursuit. Since graphs often have subgraphs that are optimized by regularity there is good reason to investigate the nature of how the simple graph network design contributes to optimization. The graph theory developed in this article directs towards likely real-world logistics application. Logistics support concerns itself with maintaining an appropriate flow of personnel and supplies along optimal paths. A need exists to represent logistics flow often with a consistent optimized network.

The planning of a complex new supply and/or maintenance system requires consideration of overall network aspects. Many alternative network design possibilities exist for locating supply points and maintenance facilities. Location and the number of roads joining the elements of a logistics system at minimal cost and maximal flow constrain network design.

The next hypothetical example illustrates the situation that gives rise to maximal logistics flow at minimal cost. Suppose the following conditions constrain a logistics network.

- S supply points are required.
- T work locations are to be serviced by S supply points.
- Each of the S supply points requires exactly E arcs (roads/paths) exiting from one point to other supply points.
- Each supply point has exactly D arcs leading from it to a select subset of depots at N sites.
- Each of the intermediary nodes has the same number of paths leading into them from the supply points.
- No unnecessary parallel roads are allowed between nodes.

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- No pointless circular roads starting at a node and terminating at the same starting point.
- The network consists of N sites servicing the work locations in order that all locations have an equal number of roads terminating at their location.
- Each of the nodes of the logistics network may be reached from any other location within the network.
- Each of the nodes has roads leading from it to the most relevant nodes required.
- The total number of distances between nodes is minimal.

The general design constraints in this sample leave a great deal of leeway for designing an optimal logistics network. If costs to build roads from each node to the others were given, then one might impose the requirement that total costs are to be minimized. Many other additional complications could be introduced. Because the purpose of this illustration is only to incite the imagination as to potential applications of network theory, the foregoing example serves well enough to accomplish that objective.

Basic Concepts and Notation

Let G be any set and F a family of subsets from G. For any element x in G define a function f(x,F,) to be the cardinality of the subfamily of subsets H in F where x∈H. f(x,F) is the frequency function of x in G relative to F. The set of all subsets of G is notated as 2(G). The cardinality of the family of all subsets G={a1,a2…an) containing any ai in G for i=12…n is 2^{n-1}.

Let S(x,F)={A∈ F| x∈A} and S(A,F)={X∈F | C(X)=C(A)}. The notation C(A) is short for cardinality of A. T(x,A) is all subsets of F of the same cardinality as A and with x∈A. We have T(x,A)=S(x,F)∩S(A,F). The following list of elementary properties hold.

[1] f(x,F)=C(S(x,F))

[2] f(x,F) is constant if and only if C(S(x,F)) has the same cardinality for all x∈

[3] f(x,F) =0 for x∈G if and only if F is the null set

[4] f(x,F)=1 for all x∈G if and only if F partitions G in separate sets of equal cardinality in F having the entry x.

[5] f(x,2(G)) has constant frequency 2^{n-1}

[6] f(x,S(A,F)=1 for A⊂F consisting of disjoint subsets of equal cardinality containing x as an element. f(x,S(A,F))=0 for x∈G not a member of any of the elements of A.

[7] For a finite set of G elements and integer r less than or equal to n where F=2(G) we have f(x,S(2(G))=(n-1)!/[(n-r)!(r-1)!]

Proof: Because there are n!/[(n-r)!r!] subsets contained in G of size r and [((n-1)!/[(n-r-1)!r!] subsets of size r not including x the conclusion follows by a simple subtraction and combining of terms. The symbols C0,C1,C2 …represent the ascending order of transfinite cardinal numbers where C0 is the cardinality of the set of integers.

[8] If C0<C(G) then f(x,S(C(G))=C(2(G)).

Proof: For the set H=G-{x} and any x in G the cardinality of all subsets of H is C(G) to the C(G) power which is C(2(G)). To each subset of H of size C(G) one can adjoin x without disturbing cardinality. The result follows from this observation.

[9] For the power set of G, f(x,2(G)) is a constant equal to 2 to the power C(G)-1

Proof: If C(G)≥C0 then the result follows from [8]. [7] and the Binomial theorem proves the finite case. The last results are the only intentional excursions into the realm of transfinite arithmetic and are only considered for aesthetic reasons. Infinite graphs are virtually an unexplored region of mathematics with little, if any known applications. Only finite cases are addressed in this article unless convenient to cover both cases, simultaneously.

[10] If A and B are families of subsets from a finite set G such that A⊂B φ≠B and f(x,A)=f(x,B) for every x∈G then A=B.

Proof: For any x϶G, let F(A)={H∈A| x∈H} and F(B) similarly defined. In general, one has F(A)⊂F(B) because A⊂B. Since C(F(A))=C(F(B)) and G is finite, it must be that F(A)=F(B) by the assumption that frequency functions agree. If H∈B there exists x∈H∈F(B)=F(A)⊂A proving B⊂A. In particular, if A is a family exhibiting frequency of 2 to the n-1 power then A must be the power set of G.

[11] For a set G of n≥2 elements there exists a family of subsets A⊂F all of size k for k=1,2…n satisfying f(x,A)=k, C(A)=n and if B is any other family with members of size k and constant frequency k, then C(A)=C(B).

Proof: Let G={a(1),a(2),…a(n)} be any ordering of G. For notational convenience assume a(i+n)=a(i) for i=1,2,…n are the same element of G and [a(i),a(i+k-1)]={a(j)∈G| i<j<(i+k-1)}. The family of subsets A={[a(i), a(i+k-1)] | i=1,2…n} has size n and f(x,A)=k for all x in G. For if x is any element of G, then x=a(j) for some j and a(j)∈[a((j-p), a(j+k-1-p)] for p-0,1,…k-1.

Now suppose B is any other family such that f(x,B)=k and H∈B implies C(H)=k. Because the total number of occurrences for all x in G has to be the same for both families A and B, the cardinality of A and B must be the same.

[12] For a finite set G of n elements and any integer m such that $0 \le m \le 2^{n-1}$ there exists a family of sets A satisfying f(x,A)=m.

Proof: In [9] it was shown that the power set 2(G) has frequency 2 to the power n-1 for any x∈G. (half of the power set has x in it and the rest do not have x because the power set is determined by counting subsets with x in an entry in 2(G) and x not in any entry of 2(G)). The family consisting of the empty set is a family whose frequency is identically zero. Now suppose A is any non-empty subfamily from 2(G). The complement of A is notated A'. The family defined as P=2(G)-{A,A'} has frequency 2 to the n-1 power minus one. In generally, if one removes k such pairs of sets with their complements, one obtains a family, P=2(G)-{A1,A1',A2,A2'…. Ak,Ak'} having frequency of 2 to the n-1 minus k. Since 2(G) has 2 to the n power elements it is clear that k can vary up to 2 to the n-1 power.

Multivalued Function Interpretation of Graphs

Notational conventions, basic concepts, and definitions explained next are from reference [2]. Generally, a graph is notated G and can be regarded as an ordered triplet (V(G),E(G),g) where V(G) is a set of points called vertices, E(G) is a set of arcs called edges, and g is an incidence function from E(G) into an unordered pair of vertices of G. The vertices need not be distinct. If e is an edge and u and v are vertices associated with e by g, then g(e)=uv and e is said to join u and v. The vertices u and v are called ends of e in this case.

A graph is simple if it has no loops or links. A loop is an edge with identical ends and a link is two distinct edges with the same ends.

The degree of a vertex v in G is the number of edges in E(G) incident with v with each loop counting as two edges. A graph G is k-regular if d(v)=k for all v∈V(G). A regular graph is one that is k-regular for some k.

A subset S of V(G) is called an independent subset of G, if no two vertices of S are adjacent in G, where adjacent means have edges to the same vertex. For a vertex v∈V(G) let A(v) be the set of all vertices adjacent to v. An independent set of vertices S is said to be maximal, if it is not contained in a larger independent set.

A multivalued function h is a function from a set D into subsets of a set R. I.e. h is a mapping from D into 2(R). A multivalued function h(v)=A(v) is of special interest in the study of graphs, especially for simple graphs.

Next results show how one can create simple regular graphs on even sets of vertices.

Every multivalued function φ on a nonempty set K into its power set 2(K) gives rise to a graph G where V(G)=K and E(G) is all edges between adjacent vertices satisfying g(e)=uv for some u,v in K. If one assumes g has no fixed points, then G has no loops and is considered a simple graph. G is p-regular if d(x)=p for all x in K.

On the other hand, if G is a p-regular graph then g(e)=xv for e∈E(G) for a pair x,v∈V(G) defines edges in G as a function g(x,v) with domain a subset of V(G)×V(G). Moreover, since G has no loops, g has no fixed points satisfying g(e)=xx for x∈V(G). Because G is p-regular we have f(x,F)=p) for the family F={A(x)| x∈V(G)}. We summarize the foregoing discussion in [13].

[13] A graph G=(V(G), E(G), g) is p-regular and simple if and only if the following holds:

i. F={A(x) | x∈V(G)} and g(e)=uv for edges in E(G)

ii. g(x) for x∈V(G) is a function on V(G) having no fixed points with g(x)=xx

iii. f(x,F)=p for all x in V(G).

We observe that [13] summarizes the set of equivalent conditions for a graph to be p-regular and simple. Even more from the derivation we see that [13] says a simple p-regular graph is equivalent to a special type of multivalued-function γ(G)=(V(G), E(G),g). The function γ characterizing graphs does not in any way depend on finiteness. Hence, [13] applies to both finite and transfinite simple p-regular graphs.

[14] If V(G)={a1,a2,…an} and p is any integer from 1 to n, then the function g(e) gives rise to a p-regular simple graph if and only if the multivalued function defined by the rule h(ai)=A(ai)'-ai (i=1,2….n) defines a simple (n-p-1)-regular graph.

Proof: ai (i=1,2…n) occurs exactly p times in the A(aj) for j=1,2…n and n-p-1 times in A(aj)'-{aj}. Therefore, h is a n-p-1 regular simple graph. Conversely, replacing A(ai) (i=1,2…n) by A(ai)'-{ai} yields the multivalued function g(ai)=(A(ai)'-{ai}=A1(ai) for i=1,2,…n), defines a (n-p-1) regular simple graph g.

[15] For any finite even set K consisting of 2n elements a1,a2…a2n there exists a p-regular simple graph G=(K,E(G),g) where p is any integer from 1 to 2n-1.

Proof: Suppose K is divided into two subsets H1={a1,a2…an} and H2={a(n+1), a(n+2)….a2n} By [11] a family F of n subsets exists from H2 with frequency k for k=1,2,…n-1 such that every Ai in F has cardinality k where F={A1,A2…An}. By introducing an edge from each ai in H1 to the elements of exactly one Ai∈F we can construct a simple p-regular graph G(p). To complete the proof one can apply [14] to construct a (2n-k-1)-regular simple graph for k=1,2…n-1. The multivalued function g(ai)=K-{ai} generates a (2n-1)-regular simple graph with vertices from K.

Graph Matrix Representations

Generally, a walk in a graph G is a finite alternating sequence of connected vertices and edges. The vertices v(i-1) and v(i) are called the ends of the edge e(i). Two vertices u,v in V(G) are connected, if there exists a walk with distinct vertices and edges from u to v. Connectedness defines an equivalence relation on V(G). This means that V(G) can be partitioned into subsets V1,V2….Vr where u and v are connected if and only if they belong to the same Vj. The subgraphs of G having the Vj as vertices are called the components of G. If G has one component, then G is said to be connected.

Every graph is the totality of all of its components. In characterizing finite simple regular graphs, it is only required to consider connected ones because the others are made of connected ones.

Graph matrix representations play an important role in the development of graph theory. The two matrices representations most frequently encountered are the adjacency and incidence matrices. The incidence matrix definition given here is different than the one found in [2] but almost the same as the one in [3] with the minor exception that only ones and zeros are allowed in [3].

Employing the usual notation for a matrix [Xij] one defines the adjacency matrix by setting Xij to 1 when g(vi,vj)=vivj=e for some edge e in E(G) and setting Xij to 0 otherwise. (note The incidence matrix [Mij] has entries Mij defined on V(G)xE(G) determined by the number of times a vertex is incident with an edge. A single loop of an edge associated with a vertex is assigned incidence value 2. Therefore, the possible entries in an incidence matrix are 0,1, and 2. Note that the incidence matrix as defined does not distinguish links but only characterizes multiple links by its assignment in the matrix as 1. Obviously, the incidence matrix conveys no new information about a known graph but offers an equivalent and convenient method of depicting the graph connectedness by edges.

We observe finite graphs with n vertices have n! equivalent adjacency matrices corresponding to the n! permutations of V(G). To characterize a simple p-regular graph in terms of these matrix representations we define and characterize a simple p-regular graph as having no 2 's in its incidence matrix, two vertices cannot have incidence with two edges, and zeros occur in the main diagonal of any of its adjacency matrix representations. In addition, the sum of any row or column in an adjacency matrix must be p.

Next, a basic decomposition related to the adjacency matrix is explained. For this decomposition one forms a descending sequence of degrees of vertices from a finite graph G. Suppose there are qi vertices of degree di for i=1,2…r. One can arrange the vertices of the graph in groups according to their degrees so that the vertices q1,q2…qn can build an adjacency matrix, where submatrices Ai of sizes qixqi are arranged so that each has its own main diagonal along the main diagonal of the adjacency matrix describing the adjacencies of graph G. Because G is a finite graph, the existence of this decomposition is clearly possible.

If G is a simple p-regular graph, then the decomposition Ai (i=1 to r) represents adjacency submatrices for simply connected p-regular components of G. Hence, if one assumes G is also connected, then G has only one component with adjacency matrix A1.

The basic decomposition for graph G offers means of analyzing the structure of G according to vertices grouped by degrees. In essence each such grouping may be viewed as an incomplete simple p-regular connected subgraph of G. This last observation can prove to be very useful when designing communications, logistics pipeline, computer hardware configuration, and other similar graph/network systems. Specifically, the arcs required to emanate from any connected grouping of vertices of like degrees to the other vertices may be viewed as a logically consistent number of arcs, which must be capable of being tied to the rest of the network coherently.

The next result follows almost immediately from basic definitions. A regular graph's structure is investigated using this next observation as a starting point. Note that the result holds for transfinite cases.

[16] For a set K, the multivalued function defined by g(x)={x}' determines a simply connected (C(K)-1)-regular graph which we denote as Z(K).

For a finite set K={a1,a2…an} one can represent the graph Z(K) as an nx(n-1) matrix where image values of g are rows. Obviously, any rearrangement of the entries within a given row yields an equivalent matrix representation of Z(K).

If G is any simply connected p-regular graph with n vertices, then its associated multivalued function g(x)=A(x) can also be represented by a matrix T similar to Z(K), but with p columns. Each element of V(G) occurs in such a matrix representation with the same frequency due to the regularity assumption. Therefore, a rearrangement of T exists in which each vertex occurs in exactly one column. Furthermore, since the rearrangement has n rows, each column of it must be a complete listing of the vertices V(G). Any such representation of a simple p-regular graph is called a standard representation.

Suppose T is a standard representation of a simple p-regular graph. For edges ai-ail and ail-ajk either ajk is in column one for every i=1,2…n or ajk is not in column one for every i=1,2…n. No mixed state can exist relative to the cycles determined by the edges. Suppose that ai-ail and ail-ajk are edges of the graph such that ail and ajk are both in column one of the standard representation T and bi-bil and bil-blm are edges of the graph such that bil is in column one and blm is not in column one. In this case, if one eliminates all edges of G having terminal vertex in column one of T, then the resulting regular graph generated represented by the diminished matrix could not be either (p-1)-regular or (p-2)-regular because the vertex ail would have degree p-2 while the vertex bil would have to have degree p-1.

On the other hand, either of the two acceptable cases might occur provided n is an even number. But if n is odd then no way exists to pair elements in cycles mapping into the first column of the matrix T. Consequently, one might conjecture that generating new regular graphs by column elimination on a standard representation of a graph having an odd number of vertices must progress by eliminating a pair of entries in the standard representation at a time and then compressing the remains of the matrix and forming a new standard matrix with 2 less columns. In fact, this is exactly what must happen. Later this assertion will be completely proven but first some preliminary results must be established.

The next two results follow quite easily by employing basic acceptable matrix operations on rows and columns yielding equivalent matrices representing Z(G).

[17] Every simply connected (n-1)-regular graph G with vertices V(G)={a1,a2…an} has a matrix representation of Z(G) of the form

a2 a3 a4…...an

a3 a4…. an a1

a4 a5..an a1 a2

………………

………………

an a1….a(n-2)

a1 a2….a(n-1)

[18] Very simple (2n-1) regular graph G with vertices V(G)={a1,a2,...an,b1b2..bn} has a matrix representing Z(G) of the form

b1 b2..........bn a2 a3.......an

b2 b3.... bn b1 a3 a4...an a1

.

.

.

bn b1...... b(n-1) a1 a2.......a(n-1)

a1 a2...........an b2 b3...........bn

a2 a3...........a1 b3 b4.......bn b1

a3 a4.......a1 a2 b4 b5....bn b1 b2

.

.

.

an a1.......a(n-1) b1 b2.......b(n-1)

We observe that the matrix of [18] may be used to prove [15]. Taking this view one could define the graph in terms of matrices.

Now assuming n is odd, one can remove all links determined by column 1 in the matrix of [17] keeping in mind that entries outside of column 1 may also have to be eliminated. The resulting new matrix symbolized as A-1 is nx(n-3) after elimination and contraction. A1 represents a simply connected (n-3)-regular graph. Of course, the process may be repeated to construct simply connected k-regular graphs for k=2,4..n-1.

For an even number of vertices one might apply the same approach to the matrix of [18] to obtain simply connected k-regular graphs for k=n,n+1,...2n-2. Since the first n columns of the matrix of [18] may be viewed as the matrix of a graph of a simply connected n-regular graph, one can eliminate columns one by one to form simply connected k-regular graphs for k=1,2...n-1, too.

It is worth noticing that the preceding discussion applies if one starts with a standard representation of any p-regular graph. Also, as observed in [14] a simple p-regular graph G has a complementary graph G'. By adjoining standard matrix representations of G and G' one obtains a matrix representation of Z(G). Now working in reverse by eliminating columns and contracting, if necessary, one can return to a standard representation of G. In essence one can view Z(G) as a unity element and G and G' as inverses relative to the unity.

The next theorem gives additional structure of the simple regular graph.

[19] If C(V(G))=2^ur and r is odd, then there exists a standard representation of Z(G) whose entries may be grouped column-wise in stacked blocks of $2^k2^{u-k}rx2^{u-k}r$ submatrices for k=1,2....u which when considered pair-wise have the same general form as the first n columns of the matrix of [18] while the last r-1 columns can be stacked in 2^ur(r-1) matrices in the general form of [17].

Proof: The proof of this result follows from a finite induction argument on u. To help the reader the result is illustrated for the case when u=2 and r=3.

V(G)={a1,a2,...a12}, u=2,r=3,C(V(G))=2^ur

*Submatrices below are all assumed to be in a standard form.

Entries from:

a7,a8,......a12 : 3x3 : 3x2 :

6x6 : a4,a5,a6 : a1,a2,a3 :

................: .... :

Entries from : 3x3 : 3x2 :

a1,a2, ...a6 : a1,a2,a3 : a4.a5.a6 :

6x6 : ....:....................:

:............:................:

: 3x3 : 3x2 :

: a7,a8,a9 : a10,a11,a12 :

................: .... :

Proof: If u is assumed to be zero, then the theorem reduces to [17]. Suppose Z(G) has 2^{s+1} p entries. Split V(G) into two subsets A={a1,a2,...a(2^sp) and B={b1,b2...b(2^sp)}. The induction assumption may be applied to A and B to form matrices M and P, respectively.

Let U be any 2^spx2^sp standard matrix with entries from B and let V be a standard form matrix with entries from A. Stacking U on top of V first and then stacking M on top of P yields a standard representation of Z(G) having the form |UM| proving the result by finite induction. |VP|

Construction of Regular Graphs

A permutation w is a bijective mapping on a set G. A permutation is said to have a cycle of length k, if there exists a subset S={a1,a2....a(k)} of G such that w(ak)=a1 and w(ai)=a(i+1) for I not equal to k. A permutation is a complete cycle, if G=S. Two permutations w1 and w2 are inverses of each other, if w1w2(x)=w2w1(x)=x

Each column in a standard representation of a regular graph may be viewed as a permutation on its vertices.

Two elements x and y are pair-wise inverted by a permutation, if w(x)=y and w(y)=x. A column of a standard representation is pair-wise inverted, if its entries consist of pair-wise inverted entries. Clearly, this can happen only when the graph has an even number of elements.

[20] A graph G is simply p-regular if and only if there exists a standard representation of G having either cyclic columns occurring in pairs that are inverses of each other or pair-wise inverted columns.

Proof: Suppose A is a standard matrix having only the two types of columns. A represents a simple graph. The pair-wise inverted columns of A, if any exist, will preserve regularity, because each such column gives rise to exactly one link for each element. Similarly, each pair of inverse columns establishes exactly two links for each element.

Suppose G has an even number of vertices. Let $A=[x_{ij}]$ b`e a standard representation of G and let $V(G)=\{a_1,a_2\ldots a_n\}$, where it is assumed V(G) is ordered as listed. The multivalued function mapping a_j into the subset $\{x_{1j}: j=1,2\ldots p\}$ for $j=1,2\ldots n$ is another way of representing G. Because the multivalued function associates a_1 with x_{11} must be equal to a_j for some j, we have x_{11} associates with a_1 within the standard representation, too. If a_1 is not in the j1th position, then one can interchange a_1 with x_{j1}. (Recall that previously it was found that for a given column of standard representation either the column is pair-wise inverted or all of the column's entries have inverse entries outside of the given column.) Now consider the proceeding a_i not already paired. If possible permute entries of row i so that the entry in the i1th position is unequal to all preceding entries handled. I.e. for a_1 and x_{11} equal to a_j. If this cannot be done, then return the matrix to the original standard representation and attempt the process on the next column. If a pair-wise inverted column x is determined by the process described, then the remaining columns can be put in a standard representation, since G is regular and x determines exactly one link for each vertex in V(G).

Suppose one has exhausted all columns that can be put in pair-wise inverted form and the remaining columns are in a standard representation. For simplicity, suppose the first t columns are pair-wise inverted. The (t+1)th column's inverse entries must all be in column t+2 through column p according to previous observations. By a procedure similar to the first process one can interchange elements so that column t and t+1 are cyclic as in conditions given in [20]. Continue this process until all columns are paired. There cannot be any columns left because otherwise such columns could be put in standard representation and the process continued. The process must terminate as it is a finite one.

If it is supposed that V(G) is odd then the immediately preceding process may be used to form pairs of columns satisfying the first condition in [20] only.

[21] If G is a simple p-regular graph having an odd number of vertices then p is even.

[22] A graph G having an odd number of vertices is simple p-regular if and only if there exists a standard representation of G having cyclic columns in pairs that are inverses of each other.

Proof: This follows immediately from [20].

Gessel [1] offers a method for counting regular graphs as a special type of Latin square. The combinatoric formula presented is quite complex, because it covers all possible Latin squares. In most applications, it is doubtful whether a knowledge of the total number of regular graphs will be helpful since restrictions on the logistics network usually confine interest to only a small subset of the total number of possible logistics networks. The numerical example given later illustrates this point [2].

[23] Every simple regular graph G may be obtained by column elimination of a standard representation of Z(G).

Proof: Let G be any simple regular graph. [14] shows how a complementary simple (n-k-1)-regular graph G' may be obtained from G. G and G' have standard representations A and B, respectively. The nx(n-1) matrix obtained by joining the columns of A and B is a representation of Z(G). The result follows immediately.

Numerical Example

Suppose one wishes to create a hypothetical logistics network consisting of six depots connected by a road system with exactly three roads accessing each of the depots so that each of the depots number 1 to 3 have a road between them and the ones numbered 4 to 6. Suppose it is known exactly how much it would cost to build a road from any one of the depots to another [3]. The cost data for building the roads between each of the roads is given in the next table. The depots will be notated as 1,2,3,4,5, and 6.

	1	2	3	4	5	6
1		20K	15	18	13	22
2	20K		12	18	15	17
3	15	12		14	12	17
4	18	18	14		16	24
5	13	15	12	16		15
6	22	16	17	24	15	

The need is to know how can the road system be built cheapest under the assumptions of the chart's costing. Here is a step by step solution to finding the optimal cost. As in [17] we form a standard representation of Z(G) shown next as a matrix.

4	5	6	2	3
5	6	4	3	1
6	4	5	1	2
1	2	3	5	6
2	3	1	6	4

: 3 1 2 4 5 :

:……………:

a) From the theory developed it is clear that two networks satisfying the conditions of the problem are obtained by eliminating either columns two or three together or columns four and five together of the standard representation Z(G). Here are the two considered possibilities described by their adjacencies as follows: A(1)={2,3,4}, A(2)={1,3,5}, A(3)={1,2,6}, A(4)={1, 5,6}, A(5)={2,6,4}, A(6)={3,4,5} describes the first network. The other network is: A(1)=(4,5,6}, A(2)={4,5,6}, A(3)={4,5,6}, A(4)={1,2,3}, A(5)={1,2,3}, A(6)={1,2,3}.

b) The last procedure is to simply hand compute cost for each of the two networks using the cost chart to see which one yields the least cost. The first one gives a total cost of 304K and the other network costs 291K based on the cost matrix. Clearly, all other factors considered equal there is a savings of 13K by picking the second choice.

c) No doubt there are a number of other possible configurations for building any network. A computer program seems like the best way to handle finding the cheapest alternative for large scale networks.

Summary and Conclusions

Several new interesting possibilities now exist because of the foregoing research effort. The structure of a simple regular graph has been made completely transparent by the results given. The results invite the computer-bent logistician to devise an efficient automated algorithm capable of examining all possible regular network systems of size n relative to specific attributes, such as cost, for example. For moderate size n the associated computer problem should be doable despite the search growing factorially as n is increased. As stated earlier most logistics networks will have regular subnetworks allowing for a fairly wide application of the theory of this article.

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Dr. Joseph E. Brierly earned a doctorate degree in mathematics from the Wayne State University in 1975. In recent years he has been involved in mathematical logistics modeling for the US Army. Before that he taught several years at the university level. Dr. Brierly is active in doing mathematical research in both pure and applied mathematics. He has recently published a two page solution to the famous Fermat's Last Problem in 2019. He is revisiting an abstract of the article that he presented for the American Mathematical Society symposium at the University of Vermont in the 1990s. The article gives a standard proof for the famous Four Color Theorem and generalizes it to the 2N Color Theorem.

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